NOTATION

T, temperature; A, affinity for conversion; ξ, internal variable; u, specific internal energy; s, specific entropy; ρ, density; J_s, entropy flux; Ψ, local scattering potential; c, specific heat; λ, thermal conductivity.

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STRONG TEMPERATURE-FIELD DISCONTINUITIES IN A NONLINEAR MEDIUM

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The conditions for the appearance of temperature discontinuities in a nonlinear medium with finite relaxation times of the heat flux are analyzed.

General laws of the appearance and propagation of simple and shock waves in heat-transfer theory were discussed in [1, 2]; the good prospects for using gasdynamic methods in modeling high-intensity thermal processes was noted.

Below, gasdynamic methods are used to investigate the wave equation of heat transfer and construct its solution behind the front of a strong temperature discontinuity.

1. Simple Heat Waves and the Appearance of a Strong Heat-Field Discontinuity

The generalized heat-transfer equations of [2, 3] form the starting point

$$cT_t + c\gamma T_{tt} = (\lambda T_x)_x, \quad (1)$$

$$q = -\lambda T_x - \gamma q_t. \quad (2)$$

If high-intensity nonsteady heat transfer occurs [1], or the thermophysical properties of the medium are such that the influence of the thermal-relaxation parameter γ is significant [4], then the quantity cT_t in Eq. (1) may be neglected. Adopting this assumption and setting $\lambda = \lambda(T)$, c, γ = const, it is found that

$$T_{tt} = \left(\frac{\lambda}{c\gamma} T_x\right)_x. \quad (3)$$

Obviously, Eq. (3) is equivalent to a system of quasilinear equations

$$T_t = R_x,\ R_t = L_x,\ L = L(T);\quad \frac{dL(T)}{dT} \equiv \dot{L}(T) = \frac{\lambda}{c\gamma} > 0. \quad (4)$$

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The relation between the auxiliary function R = R(x, t) and the heat flux is found by integrating Eq. (2):

$$q(x, t)=\beta^{-1}[q_0(x)-\beta\gamma cR+c\int\beta R dt],\quad \beta(t)=\exp\left(\frac{t}{\gamma}\right). \tag{5}$$

The system in Eq. (4) coincides in form with the equations of plane one-dimensional nonsteady isoentropic gas flow in Lagrangian coordinates [5]:

$$v_t=u_\psi,\quad u_t=-p_\psi,\quad p=p(v). \tag{6}$$

Hence, the Cartesian coordinate x, temperature T, and the functions R(x, t), $-L(t)$ are analogous, respectively, to the mass Lagrangian coordinate ψ, the specific volume v, velocity u, and pressure p(v) in a flux with constant entropy.

Writing the Riemann invariant

$$r=R+\int_{T^0}^{T}\ddot{L}^{\frac{1}{2}}dT,\quad s=R-\int_{T^0}^{T}\ddot{L}^{\frac{1}{2}}dT$$

a simple heat wave for which one of the invariants is constant – e.g., $r = r_0$ = const – is considered. Then the s characteristics are straight lines in the plane x, t [5], such that the constants s, T, R, and L satisfy the relations

$$\frac{dx}{dt}=\frac{x-x_0}{t-t_0}=-\xi(r_0-s),\quad \frac{d\xi(r-s)}{d(r-s)}=-\frac{\dddot{L}}{4\ddot{L}},\quad \ddot{L}(T)>0. \tag{7}$$

The influence of the thermodynamic properties of the medium on the behavior of the characteristics in a simple heat wave is now analyzed.

(I) $\ddot{L} = \dot{\lambda}/c\gamma < 0$. The thermal conductivity of the medium is a monotonically decreasing function of the temperature. This case corresponds to the equation of state of a gas p = p(v) such that $p_v < 0$, $p_{vv} > 0$.

(a) According to Eq. (7), when $s_x > 0$, $T_t > 0$, a diverging bundle of characteristics is obtained – a heating wave, the analog of a rarefaction wave in a gas.

(b) when $s_x < 0$, $T_t < 0$, a converging bundle of characteristics is obtained – a cooling wave, the analog of a compression wave in a gas.

(II) $\ddot{L} = \dot{\lambda}/c\gamma > 0$. The thermal conductivity of the medium is a monotonically increasing function of the temperature. This case corresponds to the equation of a state of a thermodynamically anomalous gas p = p(v) such that $p_v < 0$, $p_{vv} < 0$ [6].

(a) When $s_x > 0$, $T_t > 0$, a converging bundle of characteristics is obtained – a heating wave, the analog of a rarefaction wave in an anomalous gas, propagating in the form of a narrow region of sharp variation in the flow parameters.

(b) when $s_x < 0$, $T_t < 0$, a diverging bundle of characteristics is obtained – a cooling wave, the analog of a rarefaction wave in a thermodynamically anomalous gas.

(III) $\ddot{L} = 0$, $\lambda \equiv$ const; the s characteristics are parallel.

Thus, for variants (Ia) and (IIb), the modulus of the temperature gradient of the medium decreases with increase in t. On the other hand, for variants (Ib) and (IIa), when the characteristic bundle diverges, the modulus of the temperature gradient increases with increase in t, and after a finite time interval $[0, t_{min}]$ the characteristics intersect, and the temperature gradient becomes infinite: a gradient catastrophe sets in [5]. When $t \geqslant t_{min} > 0$, the solution of Eq. (3) becomes discontinuous. Calculation by means of Eq. (7) gives

$$t_{min}^{-1}=\max_{(x)}\left[\frac{-\ddot{\lambda}(T_0)}{\dot{\lambda}(T_0)}\frac{ds_0}{dx}\right],\quad T_0=T(x, 0),\ s_0=s(x, 0),\ r=r_0=\text{const}.$$

If $t_{min}\leqslant 0$, the temperature field remains continuous when t > 0.

Note [4, 7] that for liquid helium, in the temperature range 1.2-2.0°K, the heat transfer is determined by an equation of the form in Eq. (3), and for the thermal conductivity there is both a region of monotonic increase $\dot{\lambda}(T)>0$, $T\in(T', T^*)$, and a region of monotonic decrease $\dot{\lambda}(T)<0$, $T\in(T^*, T'')$, $T'<T''$, $\dot{\lambda}(T^*)=0$. The values of T', T*, T" and also a table of values of the thermal conductivity and graphs of λ(T) obtained on the basis of [8, 9] are given in [4].

It is also known that for the given temperature range in liquid helium, it is possible for a strong discontinuity to appear in the parameters describing the thermal process in the secondary sound wave. Physical analysis of this phenomenon and a detailed bibliography may be found in [7, 10], for example.

If in the range (T', T*) the thermal process (simple-wave conditions) occurs with $t\in[0, t_1)$, $t_1>t_{\min}$, continuous transition from temperature region (T', T*) to (T*, T") is impossible: consideration of variants (I) and (II) shows that, under these conditions in liquid helium, a strong discontinuity in the temperature field appears: a secondary sonic shock wave. The same conclusion holds on passing from interval (T*, T") to lower temperatures $T\in(T', T^*)$.

2. Condition at a Strong Heat-Field Discontinuity

For a one-dimensional thermal process with plane symmetry, the integral energy-conservation law is written in the form

$$\int\limits_{x_1}^{x_2} (\rho\varepsilon|_{t_1}^{t_2})\, dx + \int\limits_{t_1}^{t_2} (q|_{x_1}^{x_2})\, dt = 0, \quad d\varepsilon = c_v dT. \tag{8}$$

Taking a closed piecewise-smooth contour C in the plane x, t, the positive direction of travel around the contour is chosen so that the region bounded by the contour is to the left of the observer. Then, with bouned and piecewise-smooth functions ρε(x, t), q(x, t) Eq. (8) is written in the form

$$\oint\limits_{C} [\rho\varepsilon dx - q dt] = 0, \tag{9}$$

which is expedient for derivation of the conditions at the line $x = x_f(t)$ of strong heat-field discontinuity. Following the algorithm of [5, 11] for constructing the conditions of dynamic matching at the strong discontinuity, Eq. (9) yields

$$\dot{x}_f \{\rho\varepsilon\} = \{q\}, \tag{10}$$

where the curly brackets denote the difference in the values of the enclosed quantity on the two sides of the discontinuity line, for example $\{q\} = q_2 - q_1$.

If ρ, c_v are constant and $\{\rho c_v\} = 0$, Eq. (10) simplifies:

$$c\dot{x}_f \{T\} = \{q\}, \quad c = \rho c_v. \tag{11}$$

Hence, in particular, it follows that, when $\{T\}\neq 0$, a zero discontinuity of the heat flux $\{q\} = 0$ is only possible for a steady discontinuity line, $x_f \equiv \text{const}$.

The condition in Eq. (10) or (11) is a consequence of the integral energy-conservation law and replaces the heat-transfer equation along the line of strong discontinuity of the thermal field.

3. Examples of Fields Including Strong Discontinuities

In a number of cases it is difficult to use well-known gasdynamic solution considered as formal mathematical solutions of Eq. (4) in thermal problems. Thus, an important class of analytical solutions obtained by hodographic transformation is formed by parametric dependences of the form, for example: x = x(T, R), t = t(T, R). It is clear that this form of solution does not permit the effective use of Eq. (5), determining the heat flux. Therefore, if the results of mathematical investigations of Eq. (6) available in the literature are to be used, it is necessary, first, to isolate the solutions suitable for the description of thermal processes on the basis of the model in Eqs. (4) and (5) and second to give these solutions a new, thermophysical interpretation.

Let

$$L(T) = -ST^{-\varkappa}, \quad \varkappa,\ S - \text{const},$$
$$\lambda(T) = lT^{-\varkappa-1}, \quad l = c\gamma\varkappa S \equiv \text{const}; \tag{12}$$

then two examples of the solution of Eqs. (4) and (5) will be given.

Example 1. Taking the solution of [5] (p. 319) as the basis, it is found that

$$T = n_1 \left[\frac{(1-\alpha)(m_1 t + m_0)}{\alpha(n_1 x + n_0)}\right]^{\alpha}, \quad \alpha = \frac{2}{\varkappa+1}, \; m_1^2(\varkappa-1) = 2m \neq 0, \; \varkappa^2 \neq 1,$$

$$q = (n_1 x + n_0)^{1-\alpha} Q(t), \; 2\varkappa S n_1^{1-\varkappa} = m(\varkappa - 1), \; \beta = \exp\left(\frac{t}{\gamma}\right), \tag{13}$$

$$\beta Q = k + mc\gamma (1-\alpha)^{\alpha-1} \alpha^{\alpha\varkappa} \int (m_1 t + m_0)^{-\alpha\varkappa} \beta dt, \quad k \equiv \text{const}.$$

There are no other constraints on the choice of the constants $\varkappa$, S, m, k, m_0, and n_0 other than those already enumerated. So as to be specific, it is assumed that $m_i > 0$, $n_i > 0$, $i = 0, 1$.

Suppose that a strong discontinuity $x = x_f(t)$ propagates against a "cold" background $T_1 = 0$, $q_1 = 0$, and the temperature and heat flux behind the front of the discontinuity is determined from Eq. (13). Then, integrating Eq. (11), it is found that

$$n_1 x_f + n_0 = n_3 \exp\left[n_1 n_2 \int_0^t QBdt\right],$$

$$n_1 n_2 c (1-\alpha)^{\alpha} = 1, \quad B(t) = \left(\frac{\alpha}{m_1 t + m_0}\right)^{\alpha}, \quad n_3 \equiv \text{const}. \tag{14}$$

Initially (t – 0), the discontinuity

$$\{T\} = n_1 \left[\frac{m_0(1-\alpha)}{n_0 \alpha}\right]^{\alpha}, \quad \{q\} = k n_0^{1-\alpha} \tag{15}$$

is at the point $x_f(0) = 0$ ($n_0 = n_3$). At t > 0, the position of the discontinuity front is determined by Eq. (14), and the thermal process develops in the region $x \in [x_b(t), x_f(t)]$, where $x_b(t)$ is an arbitrary function. At the boundaries of this region, T, q are calculated from Eq. (13).

If $m_0 = n_0 = 0$ in Eq. (13), then the discontinuity in Eq. (15) is zero at t = 0; if the temperature is to be finite at $x \in [x_b, x_f]$, $t \geqslant 0$, it is necessary to take $-1 < \varkappa < 1$, $\varkappa + 1 > 2\gamma$, and the function $x_b(t)$ must satisfy the condition $x_b/t^{\alpha\gamma} \to 1$ as $t \to 0$.

Example 2. The first equation in Eq. (4) allows the new variable z = z(x, t) to be introduced such that

$$dz = Tdx + Rdt.$$

After passing to the plane z, t, Eq. (4) takes a form coinciding with the equations of isentropic gas flow in Euler variables

$$\theta_t + R\theta_z + \theta R_z = 0, \quad R_t + RR_z - \theta^{-1} L_z = 0, \quad L = L(T), \; T\theta = 1, \tag{16}$$

$$x = \int \frac{dz}{T(0, z)} - \int \frac{R}{T} dt.$$

For the case in Eq. (12), Eq. (16) has the simple solution

$$T = a\left(\frac{z}{t}\right)^{\delta}, \quad R = \alpha \frac{z}{t}, \quad a = \left(\frac{1-\alpha}{\varkappa S}\right)^{\delta}, \quad \delta = \frac{2}{1-\varkappa}, \tag{17}$$

$$q = zQ(t), \quad x = \frac{\alpha t}{a}\left(\frac{z}{t}\right)^{\frac{1}{1-\alpha}}, \quad \alpha = \frac{2}{1+\varkappa}, \quad \varkappa^2 \neq 1,$$

$$\beta Q = \frac{2c(1-\alpha)^2}{\varkappa - 1} t^{-\alpha} \int \beta t^{\alpha\varkappa} dt, \quad \beta = \exp\left(\frac{t}{\gamma}\right), \quad z_b = z_b(t),$$

$$z_f = ht^{-\delta} \exp\left[\frac{1}{\alpha c}\int Qdt\right], \quad h \neq 0, \; x_b = x(z_b, t), \; x_f = x(z_f, t).$$

The thermal process is considered in the region $x \in [x_b, x_f]$, $t \geqslant 0$. If the temperature is to be finite, it is necessary to take $-1 < \varkappa < 1$; the function $z_b(t)$ should satisfy the condition

$z_b/t^{\alpha} \to h,\ t \to 0,$ but is otherwise arbitrary. The law of motion of the left-hand boundary $x_b(t)$ and the temperature $T_b(t)$ at the boundary are interrelated and depend on the choice of $z_b(t)$. When t = 0

$$x_f(0) = x_b(0) = \frac{(1-\alpha)}{a} h^{\frac{1}{1-\alpha}},\quad \{T\} = 0,\ \{q\} = 0;$$

when t > 0, the discontinuity line $x = x_f(t)$ propagates against a "cold" background $T_1 = 0$, $q_1 = 0$, and the thermal conditions behind the discontinuity front are characterized by the relations $\{T\} = T(z_f, t)$, $\{q\} = q(z_f, t)$, in accordance with Eq. (17).

NOTATION

T, temperature; x, Cartesian coordinate; t, time; c_v, specific heat at constant volume; ρ, density; γ, relaxation time of heat flux; λ, thermal conductivity; ε, specific internal energy; q, heat flux; L(T), auxiliary function; ψ, mass Lagrangian coordinate; u, gas velocity; v, specific volume of gas; p, pressure; $x = x_f(t)$, line of strong discontinuity; $x = x_b(t)$, boundary of one-dimensional region. Indices: an independent variable as a subscript denotes partial differentiation; a dot over a function denotes differentiation of a function of a single argument.

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